# Fracture stress obtained from the elastic crack tip enclave model

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The fracture stress of an elastic–plastic solid is calculated by making an approximate analysis of the crack model in which a small enclave that is surrounded by a plastic region is considered to exist at the crack tip. An attempt is made to improve Thomson's earlier analysis of this crack model by ensuring that displacements as well as traction stresses are continuous across elastic–plastic boundaries and by ensuring that the correct fracture equation is obtained for the limiting case of a perfectly elastic and a perfectly plastic solid. The fracture stress is found to increase if either or both the yield stress of the material is lowered or the rate of plastic work-hardening is reduced. It is found that the fracture stress, in contrast to Thomson's result, it is always proportional to the square root of the true surface energy of the solid.

## 1. Introduction

Thomson [1, 2] obtained a fracture equation through use of a crack model in which an elastic enclave exists at a crack tip. He found that the fracture stress depends upon the true surface energy of a solid but generally it is not proportional to the square root of the true surface energy. On the other hand, we found [3] independently using the same crack model that the fracture stress is always proportional to the square root of the true surface energy.

The fracture stress equation that we obtained is

$$\sigma_f = [4\mu\gamma/a(\pi\alpha - 2\beta)]^{1/2} \tag{1}$$

where $\sigma_f$ is the fracture stress, $a$ is the crack half-length, $\mu$ is the shear modulus, $\gamma$ is the surface energy, $\alpha = 1 - \nu$ where $\nu$ is Poisson's ratio, and $\beta$ is a term which is a measure of the plastic work that is done when a crack extends by a unit distance. (In a perfectly brittle solid $\beta = 0$; if the plastic work done is exactly equal to the loss of stored elastic energy then $\beta = \pi\alpha/2$ and the fracture stress has an infinite value.)

Thomson obtained his fracture equation by taking the known solution [4] of the stresses in the plastic zone around a stationary Mode III crack. He then matched stresses across elastic–plastic boundaries, No attempt was made to match displacements across these boundaries. We suggested [3] that the failure to match displacements across the boundary might be the origin of the discrepancy between our results. However, there is another, more serious difficulty with Thomson's solution. He based his approximate analysis on the solution of a stationary, rather than a growing crack. But from the general, exact stress and displacement field solution of a stationary Mode III it is possible to show (using Equation 21 of [4]) that if there is an elastic enclave at the crack tip that the crack actually will start to propagate when the applied stress is equal to the fracture stress of a perfectly brittle solid. (This fact served as the starting point of a fatigue crack growth theory we recently developed [5].)

In this paper we attempt to improve Thomson's analysis by using the (not completely developed) plastic zone solution of Amazigo and Hutchinson [6] for a growing crack and by matching displacements as well as stresses across elastic–plastic boundaries.


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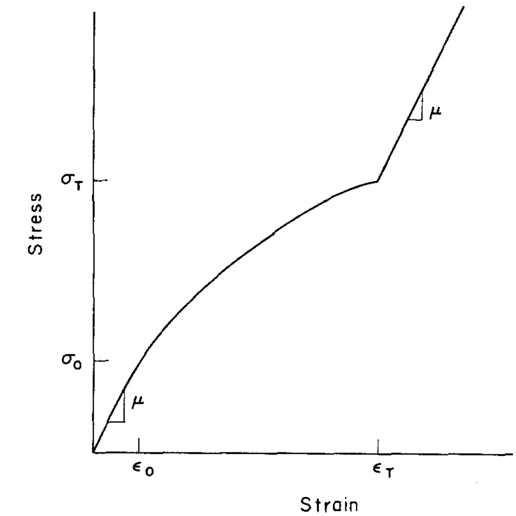


*Figure 1* Elastic–plastic stress strain curve.

## 2. Theory

In the derivation of Equation 1 it was assumed that the plastic flow curve is of the general form shown in Fig. 1. The material is elastic up to the yield stress $\sigma_0$. It is plastic above the yield stress $\sigma_0$ and below a stress $\sigma_T$. And the material is again elastic once the stress has exceeded the stress $\sigma_T$. A reasonable upper limit on the value of the stress term $\sigma_T$ is the theoretical strength of the material ($\sigma_T \approx \mu/10$); a reasonable lower limit is the average internal stress that exists in a solid when the dislocation cell size is the smallest ever observed ($\sigma_T \approx \mu/400$). Within the elastic enclave (see Fig. 2) at the crack tip the stress level is considered to exceed the stress $\sigma_T$.

Now consider matching stresses and displacements at both the boundary between the elastic enclave and the plastic zone around it and the boundary between the plastic zone and the much larger elastic region which surrounds the plastic zone. It is unfortunate that the complete, exact solution for this and similar problems of a growing crack does not exist. Thus only a rough approximation can be made. Let $r_c$ be the average dimension of the elastic enclave (which is shown circular in form in Fig. 2 although in actuality it probably is not) and $r_0$ be the average dimension of the plastic zone at the crack tip. Amazigo and Hutchinson [6] have analysed a growing crack in a linear work-hardening material with a plastic flow law shown in Fig. 3 and by the equations

$$\sigma = \mu\epsilon \qquad (2a)$$

for $\sigma < \sigma_0$ and

$$\sigma = \mu\epsilon_0 + \mu_p(\epsilon - \epsilon_0) \qquad (2b)$$

for $\sigma > \sigma_0$ where $\epsilon$ is the strain, $\epsilon_0 = \sigma_0/\mu$, and

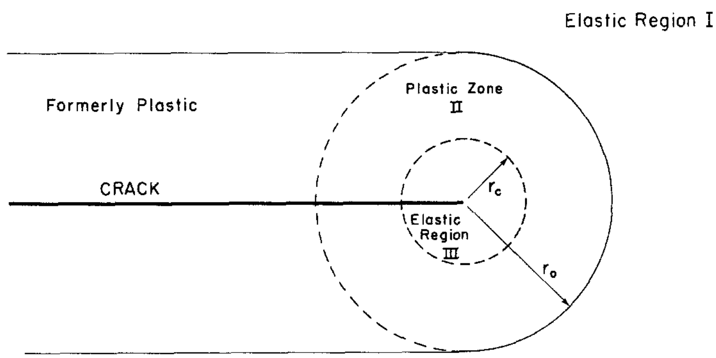


*Figure 2* Elastic regions and plastic zone around a growing crack.

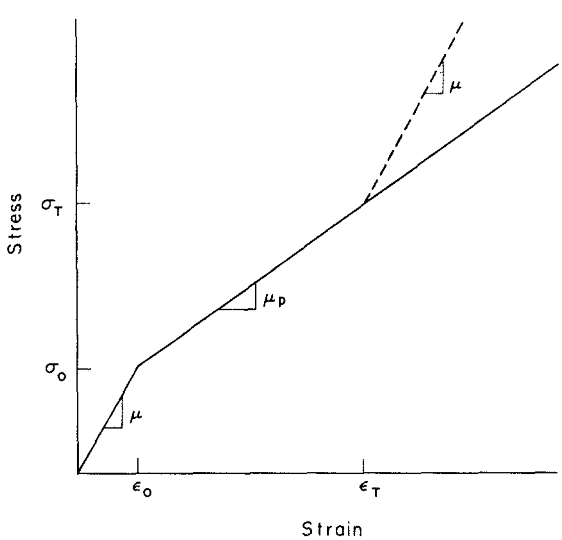


*Figure 3* Linear elastic–plastic stress strain curve.

TABLE I Values of the constant $p$ for various values of the ratio $f = \mu_p/\mu$ calculated numerically by Amazigo and Hutchinson [6] for plane strain conditions with $\nu = 1/3$

| $f$ | $p$ |
|---|---|
| 1 | 0.5 |
| 0.5 | 0.442 |
| 0.3 | 0.373 |
| 0.1 | 0.197 |
| 0.05 | 0.136 |
| 0.01 | 0.0887 |

$\mu_p$ is the linear plastic hardening rate. They showed that the strain in the plastic zone (region II of Fig. 2) is proportional to $r^{-p}$ where $r$ is distance from the crack tip. The constant $p$ is a function of the ratio $f = \mu_p/\mu$. Table I gives values of $p$ they calculated numerically for different values of the ratio. Thus within the plastic zone the strain $\epsilon_{II}$, the stress $\sigma_{II}$, and the displacement $u_{II}$ must be of the order of

$$\epsilon_{II} = (\sigma_0/\mu)(r_0/r)^p \tag{3}$$

$$\sigma_{II} = \sigma_0[1 - f + f(r_0/r)^p] \tag{4}$$

and

$$u_{II} = (\sigma_0/\mu)[r/(1-p)](r_0/r)^p + h(\sigma_0/\mu)r_c, \tag{5}$$

where $r_0$, $r_c$ and $h$ are unknown constants. The azimuthal angular dependence is ignored in this approximate treatment and the fact that the stress, strain and displacement actually consist of various components $\tau_{ij}$, $\epsilon_{ij}$, and $u_i$. At $r = r_0$ the stress $\sigma_{II}$, as it must, is equal to $\sigma_0$. At $r = r_c$ the stress $\sigma_{II}$ must be equal to $\sigma_T$. Hence $r_c$ has the value

$$r_c = g r_0 \tag{6a}$$

where

$$g = f^{1/p}/[(\sigma_T/\sigma_0) - 1 + f]^{1/p}. \tag{6b}$$

If $\sigma_T = \sigma_0$ the constant $g = 1$ and, as expected, there is no plastic zone.

In the elastic region that surrounds the plastic zone (region I of (Fig. 2) the stress must fall off as $r^{-1/2}$ at distance small compared with the crack length but large compared with $r_0$. The following general function for the stress satisfies this condition as well as the static equilibrium equations

$$\sigma_I = K/(2\pi r)^{1/2} + \sum_{m=1}^{\infty} \{K_m^*/(2\pi r)^{1/2}\}(r_0/r)^m \tag{7}$$

where $K_m^*$ are unknown constants. (Again, the azimuthal angle dependency has been ignored in each of these terms.) The stress intensity factor $K$ in Equation 7 is equal to

$$K = \sigma_a(\pi a)^{1/2}, \tag{8}$$

where $\sigma_a$ is the applied stress.

In the elastic enclave (region III of Fig. 2) the general equation for the stress is given by

$$\sigma_{III} = K_t/(2\pi r)^{1/2} + \sum_{m=1}^{\infty} \{K_m'/(2\pi r)^{1/2}\}(r/r_c)^m \tag{9}$$

where $K_t$ and $K_m'$ are unknown constants. The term $K_t$ is the "true" stress intensity factor of [3]. The form of Equation 9 ensures that the displacement at $r = 0$ is equal to zero. (Again, the azimuthal angle dependence of each of the terms has been ignored.)

To obtain the values of all the constants in Equations 7 and 9 actually requires that the angular dependence of these terms be taken into account. The traction stresses and the displacements must be continuous across the I/II elastic plastic boundary of Fig. 2 and the II/III elastic plastic boundary where I is the outer elastic region, III is the inner elastic region, and II is the plastic zone. For our rough approximation it is only necessary to retain sufficient number of terms to ensure that the stresses and displacements are continuous across the boundary when there is no azimuthal angular dependence and to ensure that the solution can reduce to that of a perfectly brittle solid when $\sigma_0 = \sigma_T$. Thus in Equation 7 retain the terms that contain the terms $K$ and $K_1^*$ and in Equation 9 the terms $K_t$ and $K_1'$.

After the stresses are set equal to $\sigma_0$ across the I/II boundary and to $\sigma_T$ across the II/III boundary and the displacements are made equal to each other across the same boundaries the following results are found.

$$r_0 = (2K^2/\pi\sigma_0^2)[1 + (1-p)^{-1} + hg]^{-2} \tag{10}$$

$$K_t = cK \tag{11}$$

where

$$c = (g^{1/2}/2)[(\sigma_T/\sigma_0) + 3p(1-p)^{-1}g^{-p} + 3h]/[1 + (1-p)^{-1} + hg]. \tag{12}$$

The constant $g$ is given by Equation 6b and $r_c$ is still equal to $g r_0$. Equation 12 can be rewritten as

$$h = \{(2cg^{-1/2})[1+(1-p)^{-1}] - (\sigma_T/\sigma_0) - (3pg^{-p})(1-p)^{-1}\}/(3-2cg^{1/2}). \quad (13)$$

When $\sigma_0/\sigma_T \ll 1$ and $f \ll 1$, Equation 13 becomes

$$h \approx (4c/3)(\sigma_T/\sigma_0)^{1/2p}(1/f)^{1/2p} - (\sigma_T/\sigma_0)(1+3p/f)/3 \quad (14)$$

which can be rewritten

$$c \approx (3/4)(\sigma_0/\sigma_T)^{1/2p} f^{1/2p} \times \{h + (1/3)(\sigma_T/\sigma_0)(1+3p/f)\}. \quad (15)$$

The value of the constant $h$ cannot be determined uniquely because actually there are more unknown constants ($h, K_t, K_1'$, etc.) than (strong) conditions to determine them all. However, enough information (additional weak conditions) is available to set sufficient limits on the value of $h$ that a good estimate can be made of the value of the constant $c$ of Equation 11. The evaluation of the constant $c$ is the primary goal of this paper.

First of all, the constant $h$ must have a functional dependence on the terms $p, f, g$, and $\sigma_T/\sigma_0$ such that in the limit of a brittle solid the constant $c$ must be equal to $c = 1$. Thus when $\sigma_0 = \sigma_T$ and $f \neq 0$ or when $f = 1$ (or when both $\sigma_0 = \sigma_T$ and $f = 1$) the constant $c = 1$. According to Equation 10 when $\sigma_0 = \sigma_T$ the term $h = (3-4p)/(1-p)$. When $f = 1$, and thus $p = 1/2$, the term $h = 2(\sigma_T/\sigma_0)/[3-2(\sigma_0/\sigma_T)]$. Thus the term $h$ could be given by an expression of the type

$$h = \{(3-4p)(\sigma_T/\sigma_0)/[1+p-2p(\sigma_0/\sigma_T)]\}^q \quad (16)$$

where the exponent $q$ could be equal to $q = 2p$, or $q = 1$, or $q = 1/2p$.

The constant $c$ must approach zero in value either when $f \to 0$ (and thus $p \to 0$) and $\sigma_0 \neq \sigma_T$ or when $\sigma_0/\sigma_T \to 0$ and $f \neq 1$ (and thus $p \neq 1/2$). (According to Rice ([7] and private conversation) the rate of plastic work of the growing crack is exactly equal to the rate of release of elastic energy under conditions like these. Hence $c$ must be equal to zero.) Thus $q \neq 1/2p$. Moreover, in the limit $\sigma_0/\sigma_T \to 0$ the elastic enclave disappears. In this situation the displacement $u_{II}$ given by Equation 5 must be equal to zero (or at least have a finite value) as $r \to 0$ and $r_c \to 0$. This is only possible if $h$ increases at a rate no faster than a first power of the ratio $\sigma_0/\sigma_T$ when $f \neq 1$. Thus when $\sigma_0/\sigma_T \ll 1$ and $f \ll 1$ Equation 15 reduces to

$$c = \eta[f(\sigma_0/\sigma_T)^{1-2p}]^{1/2p} \quad (17)$$

where the constant $\eta = (1/4)(1+3p/f)$ if $h$ increases at a smaller rate than a first power of the ratio $\sigma_T/\sigma_0$ and $\eta \approx (3/4)[3+(1/3)(1+3p/f)]$ if the increase is at a first power to this ratio.

The fracture stress equation is obtained by equating $K_t$ with $K_{cb}$ where $K_{cb}$ is the critical stress intensity factor of a perfectly brittle solid of modulus $\mu$ and Poisson's ratio $\nu$. The value of $K_{cb}$ is equal to

$$K_{cb} = (4\mu\gamma/\alpha)^{1/2}. \quad (18)$$

The fracture stress $\sigma_f$, found by equating Equations 14 and 18, is

$$\sigma_f = c^{-1}(4\mu\gamma/\pi\alpha a)^{1/2}. \quad (19)$$

For a perfectly brittle solid $c = 1$. When $c$ is given by Equation 17 the fracture stress is equal to

$$\sigma_f = \eta^{-1}[f^{-1}(\sigma_T/\sigma_0)^{1-2p}]^{1/2p}(4\mu\gamma/\pi\alpha a)^{1/2}. \quad (20)$$

The constant $\beta$ of Equation 1 is equal to

$$\beta = (\pi\alpha/2)(1-c^2). \quad (21)$$

When $c$ is small the fracture stress is very large compared with the fracture stress of a perfectly brittle solid. The conventional critical stress intensity factor $K_c$ of the solid is equal to $K_c = c^{-1}K_{cb}$ and is also large when $c$ is small.

It should be noted that the fracture stress given by Equation 19 is proportional to $\gamma^{1/2}$, a result that is in disagreement with that of Thomson [1, 2]. The fracture stress, as expected, increases when the yield stress decreases and when the rate of work-hardening decreases.

Suppose the flow stress is given by a power law relationship of the type that Equation 2a is still valid when $\sigma \leqslant \sigma_0$ but Equation 2b is replaced with the equation

$$\sigma = \sigma_0(\epsilon/\epsilon_0)^n \quad (22)$$

when $\sigma \geqslant \sigma_0$. Here $n$ is a constant with a typical value of around $n \approx 0.2$ to 0.3. In this situation Equation 2b can be approximated to Equation 22 by setting $f$ equal to

$$f = [(\sigma_T/\sigma_0)-1]/[(\sigma_T/\sigma_0)^{1/n}-1]. \quad (23)$$

The use of Equation 22 for $f$ ensures that both the power law flow and the linear flow law which is an approximation to it have common points at $\sigma = \sigma_0$ and at $\sigma = \sigma_T$. If $f$ given by Equation 23 is inserted into the previous equations of the text the approximate value of the fracture stress for power law hardening material can be found. In particular, when $\sigma_0/\sigma_T \ll 1$ and $n \neq 1$ the constant $c$ is equal to

$$c = \eta(\sigma_0/\sigma_T)^{\{(1-n)/n+(1-2p)\}/2p} \quad (24)$$

and the fracture stress is equal to

$$\sigma_f = \eta^{-1}(\sigma_T/\sigma_0)^{\{(1-n)/n+(1-2p)\}/2p}(4\mu\gamma/\pi\alpha a)^{1/2}. \quad (25)$$

The smaller the value of $n$ and the smaller the yield stress $\sigma_0$, the larger, as expected, is the fracture stress. The fracture stress is infinite in value when $n = 0$.

It is interesting to note that if the analysis of this paper were repeated using the plastic zone solution for a stationary crack that the equivalent term $h$ is always uniquely specified by the condition that $c$ must equal $c = 1$ for all values of the ratio $\sigma_T/\sigma_0$ and the exponent $n$. The value of $h$ so found gives the result that the displacement at $r_c$ is indeed equal to zero in the limit of $r_c \rightarrow 0$.

## 3. Conclusions

The approximate treatment of the crack tip, elastic enclave model given in this paper indicates that the fracture stress is always proportional to the square root of the true surface energy of a solid. This result differs from that found by Thomson. The fracture stress is found to increase if the yield stress of the solid is decreased or if the rate of plastic work-hardening is lowered.

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